A Note on the Asymptotic Properties of the Two-Sector Robinson-Solow-Srinivasan Model

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Abstract

I show that the periodic and chaotic behavior exhibited by the two-sector Robinson-Solow-Srinivasan (RSS) model in discrete-time is asymptotically irrelevant. If the discrete time interval is smaller than a critical limit, the qualitative properties of the model are the same as those in the continuous-time model.

Keywords: RSS, Stiglitz plan, Two sector, Asymptotic

JEL Classification: D90 , C62 , O41

Declaration

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1 Introduction

The choice between discrete and continuous time in macroeconomic modeling is usually seen as one of practicality, with qualitative results in one passing easily through to the other. This complacency has occasionally been challenged, for example by Foley (1975). More recently, results on a simple two-sector model have brought renewed concerns to the issue. In this note I show that these concerns are unwarranted in that a sufficiently fine discretization of time leads to the same qualitative behavior as the continuous model.

The problem of optimal growth in the RSS model was first studied in the late sixties and early seventies by Robinson (1960, 1969), Okishio (1966), and Stiglitz (1968, 1970, 1973). Stiglitz showed that in a setting of i) continuous time ii) linear utility and iii) constant discounting, the optimal capital plan monotonically converges to a steady-state level of capital. Robinson criticized all three of these assumptions, but it had to wait until 2000 when Solow (2000) revisited the model asking for a solution to the "Ramsey problem for this model" that real progress began to be made in these directions.

In particular the discrete-time version of the model, both with and without discounting, led to some surprising results. In discrete-time it is possible to find oscillating, periodic and even chaotic optimal plans for capital accumulation, while maintaining the assumption of linear utility. These results contrast with the continuous time monotonically converging optimal plans, or "Stiglitz plans". That such different behavior between the two settings is possible is of obvious concern to the macro-economist who otherwise has little reason to prefer one choice over the other. Metcalf (2008) states that, "switching from continuous to discrete time a policy can go from optimal to the worst possible result" while Khan and Mitra (2006) point to "a dramatic difference between the discrete- and continuous-time analyses" where "it is not apparent how qualitative properties of the discrete-time dynamics can be interpreted as valid approximations to those obtained in continuous-time". They state that the asymptotic implementation of Stiglitz' results for the RSS model remains yet to be accomplished.

In this note I accomplish this asymptotic result and show that, correctly formulated, there is no tension between the continuous and discrete time models. I find that when time is discretized into intervals smaller than a critical limit, determined by the parameters of the model, the possibility of non-Stiglitz behavior disappears. This limit applies in both the discounted and undiscounted settings and therefore provides a very satisfactory resolution to the concerns arising from the discrete-time results. Furthermore, the critical limit for the size of the discrete time period has an economic interpretation as the time required to build the steady-state capital stock if the entire workforce is employed only to build capital. For the US the capital stock is estimated to be around 3-4 times GDP, so any discretization capable of generating non-Stiglitz type behavior must be larger than 3-4 years. Periodic or chaotic behavior requires a discretization twice this length, or 6-8 years. The discretization refers to the period of time for which the allocation of labor between consumption and capital creation remains fixed. As far as I am aware there is no economic justification for making such an extreme assumption about the discretization of time, and therefore the implications of models for which such an assumption is critical are unlikely to offer much economic insight.

2 The Two-Sector RSS Model

The RSS model is a capital growth model that is both simple, and rich enough to allow for a variety of optimal dynamics. It also gives rise to an enlightening geometric interpretation that I will make use of in this note.

In the two-sector RSS model, one sector produces a single, infinitely divisible consumption good while the other sector produces a single, infinitely divisible type of machine. The consumption good is produced through a Leontief production function with labor and machines as inputs. One unit of labor, combined with one unit of machines produce one unit of consumption good. Machines are produced using labor only, with $a > 0$ units of labor required to produce one unit of machines. Machines depreciate at a rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available for production during each time period $t \in \mathbb{N}$.

The transition possibility set, Ω, characterizes the admissible production plans (x, x'), where x represents the amount of machines available this period, and x' the amount of machines available in the next period. From the specification of production above this set is

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}$$

The first inequality says that investment is irreversible, you cannot reduce the number of machines below those that are passed on (after depreciation) from the previous period. The second inequality says that the number of machines cannot be increased beyond putting the full one unit of labor to work building machines. For any $(x, x') \in \Omega$, the set of possible consumption good production is defined by the correspondence $\Lambda : \Omega \to \mathbb{R}_+$ where

$$\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a(x' - (1-d)x)\}$$

Here the two inequalities reflect the Leontief production function, that is production of the consumption good is limited by both the number of machines and the labor available to work in the consumption sector. Welfare in each period is linear in the consumption good so that the reduced form felicity function is $u : \Omega \to \mathbb{R}_+$

$$u(x, x') = max\{y \in \Lambda(x, x')\}$$

The discount factor, $\rho \in (0, 1]$, covers both the discounted and undiscounted cases. A program starting at x_0 is simply a sequence $\{x(t)\}_{t=0}^{\infty}$ that satisfies $x(0) = x_0$ and the technology constraint $(x(t), x(t+1)) \in \Omega$ for all t. A program $\{x^*(t)\}_{t=0}^{\infty}$ starting at x_0 is said to be *optimal* if there does not exist any other program $\{x(t)\}_{t=0}^{\infty}$ starting at x_0 that overtakes it, which is to say there does not exist an $\varepsilon > 0$, and time period t_ε such that

$$\sum_{t=0}^{T} \rho^t \Big(u\big(x(t), x(t+1)\big) - u\big(x^*(t), x^*(t+1)\big) \Big) \geq \varepsilon \qquad \forall\, T \geq t_\varepsilon \tag{1}$$

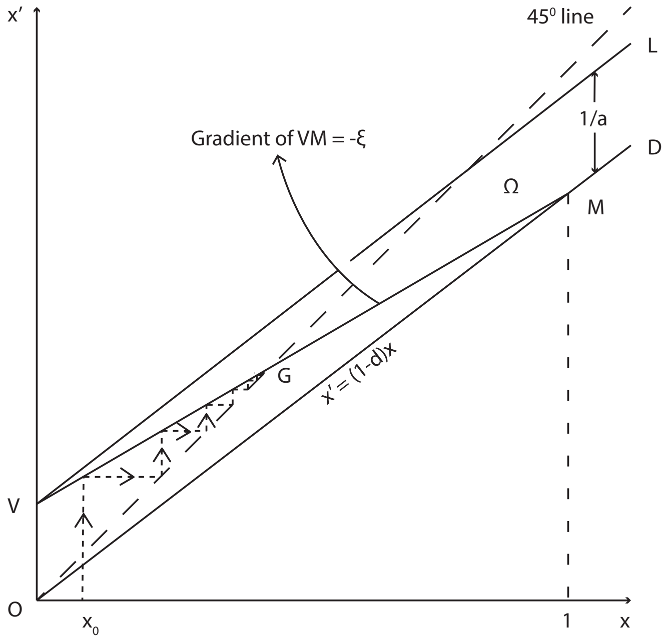

Fig. 1 When $\xi < 0$ the optimal path for the capital stock converges monotonically to a steady state. Here the capital stock is shown increasing to G from x_0. If x_0 were above G, then the capital level would decrease monotonically to G.

In the discounted case ($\rho < 1$) the sum $\sum_{t=0}^{T} \rho^t u\big(x(t), x(t+1)\big)$ converges and the optimality condition above is equivalent to the usual maximization of discounted lifetime utility.

3 The Geometry of the Model

With such a set up the two-sector RSS model is completely specified by the three parameters (a, d, ρ). While the geometry presented here is not necessary for the main result, it helps in building intuition behind what is driving the discrete-time behavior, and why it collapses for small discrete-time intervals. Here I will furnish the geometric setting without discounting ($\rho = 1$), but similar geometric diagrams can be shown for the discounted setting too. The key parameter of interest in either case is

$$\xi = 1/a - (1 - d)$$

which represents the marginal rate of transformation of capital today into that of tomorrow. The line $x' = \frac{1}{a} - \xi x$ represents the points where all machines are manned with labor, and any additional labor is used to produce more machines. Figures 1-3 show how changing the value of ξ affects the dynamics of the optimal program (these figures are based on those in Khan and Mitra (2006, 2007) which also contain more details on the optimal programs). Missing in this previous work is the crucial relationship between ξ and the size of discretization, which will be examined in section 4.

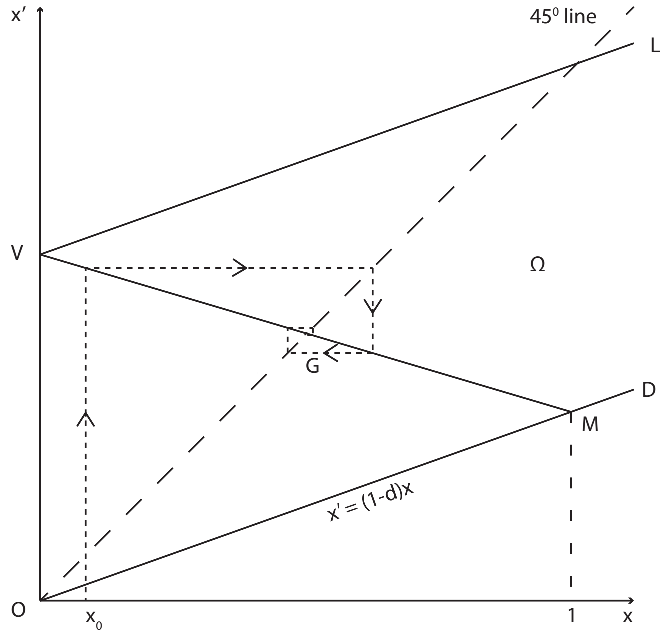

Fig. 2 When $0 < \xi < 1$ the optimal path for the capital stock converges a steady state following an oscillating path.

Figure 1 shows the behavior for $\xi < 0$. The technology possibility set, Ω, is represented by the area LVOD. The slope of the line VM is equal to $-\xi$, which in this case is a positively sloping line that crosses the 45 degree line at G, the golden rule capital level. The line VM also represents the points where full use is being made of all capital and labor. Here the optimal path monotonically converges towards the golden rule stock of capital. These types of programs are described as "Stiglitz plans".

Figure 2 begins to show some of the less expected behavior of the model in discrete time. Here, $0 < \xi < 1$, and an optimal plan, while still converging towards the golden rule stock, does so while oscillating around it.

Furthermore, figure 3 shows that convergence to the golden rule stock is not assured in the discrete time model. Here, with $\xi = 1$, the optimal path is periodic with a cycle of length 2.

Further investigation shows surprising results for $\xi > 1$. The undiscounted case can lead to overbuilding even in a capital rich economy, while the discounted case can result in chaotic behavior. See Sousa (2013) and Khan and Mitra (2005) for an exploration into the possibility of chaos in the RSS model.

For the purposes of this note, I will only require one lemma from these results:

Lemma: For an RSS model parametized by (a, d, ρ), if $\xi < 0$ then there is a unique optimal program that converges monotonically to a steady state.

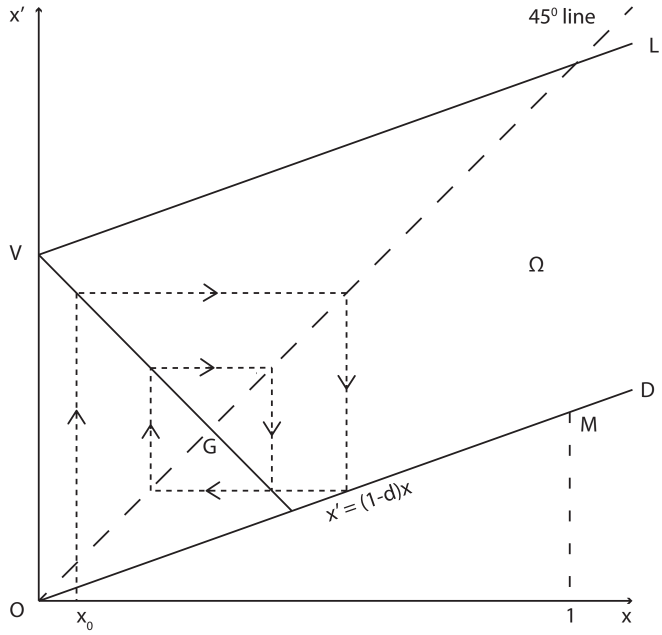

Fig. 3 When $\xi = 1$ the optimal path for the capital stock is periodic around the golden rule capital stock.

Proof: See Khan and Mitra (2006), section 5 for the discounted case. See Khan and Mitra (2007), section 5 for the undiscounted case.

4 An Asymptotic Result

The crux of this note is the reconciliation the qualitative differences between the discrete- and continuous-time models. I do this here by showing that the only relevant case for finely discretized time is the $\xi < 0$ case. Therefore, in the limit as the discrete time interval goes to zero, the only optimal plans are Stiglitz plans.

In the model presented above, during one time period it required a units of labor to produce one unit of machines, and the stock of machines depreciated by a fraction d. Interpreting this as an approximation of a continuous time technology, if we were to run the same technology over a shorter length of time, Δt, then it would require $a/\Delta t$ units of labor to produce one machine, and the stock of machines would depreciate by a fraction $1 - (1-d)^{\Delta t} \approx d\Delta t$. Using this new time period, Δt, the relevant key parameter is $\xi_{\Delta t}$ where

$$\xi_{\Delta t} = \Delta t/a - (1 - d\Delta t) = -1 + (1/a + d)\Delta t$$

and $\xi_{\Delta t} \to -1$ as $\Delta t \to 0$.

From the theory above we know that for $\xi_{\Delta t} < 0$ the optimal plan is a Stiglitz plan. Rearranging we get the following result:

Theorem: For a discretization of time such that $\Delta t < \frac{a}{1+ad}$*, the optimal plan is a*

Stiglitz plan.

Proof: $\Delta t < \frac{a}{1+ad} \implies \xi_{\Delta t} < 0$. The lemma from section 3 completes the proof.

Hence the asymptotic qualitative behavior of the discrete time model agrees with that of the continuous time model.

5 Interpretation of Condition

The condition $\xi_{\Delta t} < 0$ can be rearranged as $\frac{\Delta t}{a} < \frac{1}{1+ad}$. This now has a nice economic interpretation. $\frac{\Delta t}{a}$ is the amount of capital that can be built in one period if all workers are building capital. $\frac{1}{1+ad}$ is the golden rule level of capital, where the line VM intersects with the 45 degree line. For a Stiglitz plan this is the steady state level of capital. Therefore, for non-Stiglitz type behavior, it must be the case that the time period is long enough that a workforce entirely dedicated to building capital can reproduce the entire capital stock in one period. For the US, estimates of the capital stock are 3-4 times GDP, suggesting that a discretization of 3-4 years is required. For the more unusual periodic or chaotic behavior, the condition $\xi_{\Delta t} < 1$ is equivalent to $\frac{\Delta t}{a} < \frac{2}{1+ad}$, so that a period of 6-8 years is required.

6 Conclusion

While it may be of some relief to most macro-economists that we are able to reconcile the discrete- and continuous-time model behavior, others may be disappointed that the motivational rug has been pulled from under the theory of the $\xi > 1$ case. The mathematical results for this case seem interesting of themselves, but cannot be justified as economics unless there is a question they are able to address. Given that a coarse discretization of several years is required to generate $\xi > 1$, it would appear hard to justify such a model.

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